

Relations among the kernels and images of Steenrod squares acting on right \mathcal{A} -modules

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Abstract

In this note, we examine the right action of the Steenrod algebra \mathcal{A} on the homology groups $H_*(BV_s, \mathbb{F}_2)$, where $V_s = \mathbb{F}_2^s$. We find a relationship between the intersection of kernels of Sq^{2^i} and the intersection of images of $Sq^{2^{i+1}-1}$, which can be generalized to arbitrary right \mathcal{A} -modules. While it is easy to show that $\bigcap_{i=0}^k \text{im } Sq^{2^{i+1}-1} \subseteq \bigcap_{i=0}^k \ker Sq^{2^i}$ for any given $k \geq 0$, the reverse inclusion need not be true. We develop the machinery of homotopy systems and null subspaces in order to address the natural question of when the reverse inclusion can be expected. In the second half of the paper, we discuss some counter-examples to the reverse inclusion, for small values of k , that exist in $H_*(BV_s, \mathbb{F}_2)$.

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1. Introduction

For $s \geq 1$, let $V_s = \mathbb{F}_2^s$, the elementary Abelian 2-group of rank s . The action of the Steenrod algebra \mathcal{A} on the cohomology of BV_s has been a topic of much study (see [16, 17, 10], for example). The problem of finding a minimal generating set for $H^*(BV_s, \mathbb{F}_2)$, known as the “hit problem,” seems currently out of reach, though there are complete answers in the cases $s \leq 4$ (see [5, 8, 9, 15]). We examine the dual problem: finding a basis of the space of \mathcal{A}^+ -annihilated elements of the homology of BV_s . An element x is “ \mathcal{A}^+ -annihilated,” if $xSq^p = 0$ for every $p > 0$. Some important work has already been done on this dual problem (e.g. [2, 6, 11]). Let Γ be the bigraded space $\{\Gamma_{s,*}\}_{s \geq 0}$, where $\Gamma_{s,*} = H_*(BV_s, \mathbb{F}_2)$. We write $\Gamma_{\mathcal{A}^+}$ for the space of \mathcal{A}^+ -annihilated elements of Γ . The problem may be phrased thus:

\mathcal{A}^+ -Annihilated Problem. *Find an \mathbb{F}_2 basis for $\Gamma_{\mathcal{A}^+}$.*

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We find the \mathcal{A}^+ -annihilated problem to be more natural than the hit problem, since $\Gamma_{\mathcal{A}^+}$ admits an algebra structure; indeed $\Gamma_{\mathcal{A}^+}$ is free as associative algebra [3]. William Singer and the author proposed to study the spaces of “partially \mathcal{A}^+ -annihilated” elements, as these may be more accessible. It was found that these spaces are also free associative algebras [4]. Recall some of the definitions and notations used in [4]. The bigraded algebra $\tilde{\Gamma} = \{\tilde{\Gamma}_{s,*}\}_{s \geq 0}$ is defined by:

$$\tilde{\Gamma}_{s,*} = \begin{cases} \mathbb{F}_2, \text{ concentrated in internal degree } 0, & \text{if } s = 0, \\ \tilde{H}_*((\mathbb{R}P^\infty)^{\wedge s}, \mathbb{F}_2), & \text{if } s \geq 1. \end{cases}$$

Observe, the homotopy equivalence $BV_s \simeq \prod^s \mathbb{R}P^\infty$ connects this definition with that of Γ given above. The Steenrod algebra acts on the right of $\tilde{\Gamma}$, and in this note, all functions that we define will be written on the right of their arguments in order to maintain consistency.

Let \mathcal{S}_k be the Hopf subalgebra of \mathcal{A} generated by $\{Sq^{2^i}\}_{i \leq k}$. For each $k \geq 0$, define the bigraded space of partially \mathcal{A}^+ -annihilated elements, $\Delta(k) \subseteq \tilde{\Gamma}$, by

$$\Delta(k) = \tilde{\Gamma}_{\mathcal{S}_k^+} = \bigcap_{i=0}^k \ker Sq^{2^i}.$$

The definition generalizes to arbitrary \mathcal{A} -modules, whether unstable or not.

Definition 1. Let M be a right \mathcal{A} -module and $k \geq 0$. Define the graded space of *partially \mathcal{A}^+ -annihilated* elements of M ,

$$\Delta_M(k) = M_{\mathcal{S}_k^+} = \bigcap_{i=0}^k \ker Sq^{2^i}.$$

Define also the graded spaces of *simultaneous spike images*,

Definition 2. Let M be a right \mathcal{A} -module and $k \geq 0$. The space of *simultaneous spike images* of M is defined by

$$I_M(k) = \bigcap_{i=0}^k \text{im } Sq^{2^{i+1}-1}.$$

When $M = \tilde{\Gamma}$, the notation $I(k)$ is used.

During the course of investigating the “ \mathcal{A}^+ -annihilated problem” in $\tilde{\Gamma}$, Singer and the author made a number of (unpublished) conjectures that we later proved false. One of the most tantalizing and longest-lived of our conjectures concerned a proposed relationship between \mathcal{S}_k^+ -annihilateds and simultaneous spike images, based on the observation,

Proposition 3. *Let M be a right \mathcal{A} -module. Then $I_M(k) \subseteq \Delta_M(k)$.*

The proof is immediate from the Adem relations: $Sq^{2^n-1}Sq^n = 0$, for all $n \geq 1$. It is natural to ask whether some version of the converse of Prop. 3 could hold in $\tilde{\Gamma}$. Observe that if $x \in \tilde{\Gamma}_{s,d}$ with $d < 2^{k+1}$, then x automatically lies in $\ker Sq^{2^k}$ due to instability conditions in $\tilde{\Gamma}_{s,*}$, so there is no reason to suppose that such x lie in the image of $Sq^{2^{k+1}-1}$. However, in view of Prop. 3, we may well ask:

Question 4. *For each integer $k \geq 0$, if the internal degree of $x \in \tilde{\Gamma}$ is at least 2^{k+1} , then is it the case that $x \in \Delta(k)$ if and only if $x \in I(k)$?*

The answer is *yes* when $k = 0$, as shown in [4]. Moreover, we had experimental evidence, including many hours of machine computation, that supported a positive answer in general. A counterexample was first found for the case $k = 2$, and subsequent work was directed at proving the case $k = 1$. It wasn't until very recently that the author produced a counterexample for $k = 1$ in bidegree $(5, 9)$, and so the answer to Question 4 has to be *no* for all $k \geq 1$. In this note, we shall provide stronger hypotheses that turn the question into a true theorem. In order to state and prove the result, we introduce (in Section 2) the machinery of *homotopy systems* on right \mathcal{A} -modules. In Sections 3 and 4, we will define a suitable homotopy system for $\tilde{\Gamma}$ and use it and some variations thereof to analyze a broad class class of \mathcal{A}^+ -annihilated elements in some important right \mathcal{A} -modules (see Thm. 15 for example). Further observations about the structure of $\Delta_M(k)$ and $I_M(k)$ are considered in Section 5. Section 6 contains some structure formulas for elements of $\Delta(1)$, which lead to an explicit element $z \in \Delta(1)_{5,9}$ that is not contained in $I(1)$ (we produce the element in Section 7).

2. Homotopy Systems

2.1. Definition of Homotopy System.

Let M be a right \mathcal{A} -module. Fix an integer $k \geq 0$. A k^{th} -order *homotopy system* on M consists of a subspace $N \subseteq M$, called a *null subspace*, which may not necessary be an \mathcal{A} -module, and for each integer $0 \leq m \leq k$, a homomorphism of vector spaces $\psi^{2^m} : M \rightarrow M$ such that:

- N is stable under ψ^{2^m} .
- If $1 \leq m \leq k$ and $\ell < 2^m$, then there is a commutative diagram,

$$\begin{array}{ccc} N & \xrightarrow{Sq^\ell} & M \\ \psi^{2^m} \downarrow & & \downarrow \psi^{2^m} \\ N & \xrightarrow{Sq^\ell} & M \end{array} \quad (1)$$

- If $0 \leq m \leq k$, then there is a *homotopy relation*,

$$\left(\psi^{2^m} Sq^{2^m} + Sq^{2^m} \psi^{2^m} \right) : N \rightarrow M = N \hookrightarrow M. \quad (2)$$

Example 5. If M is considered as a chain complex with differential Sq^1 , and $N = M$, then a zeroth-order homotopy system on M consists of a chain homotopy ψ^1 from the identity map to the zero map.

Remark 6. The null subspace N is part of the definition, and there is no requirement that N be the *maximal* subspace such that Eqns. (1), (2) are valid with respect to it. In fact, if $\{\psi^{2^m}\}_{m \leq k}$ is a k^{th} -order homotopy system on M with null subspace N , then for any $j \leq k$, the set of maps $\{\psi^{2^m}\}_{m \leq j}$ is a j^{th} -order homotopy system on M with the same null subspace N , although in practice we might be able to find a larger null subspace.

2.2. The Main Theorem.

Theorem 7. *Let M be a right \mathcal{A} -module and fix an integer $k \geq 0$. If there is a k^{th} -order homotopy system $\{\psi^{2^i}\}_{i \leq k}$ on M with associated null subspace N , then*

$$N \cap \Delta_M(k) = N \cap I_M(k).$$

Moreover, for $x \in N \cap \Delta_M(k)$ and for each $i \leq k$, the element

$$y_i = x\psi^{2^i}\psi^{2^{i-1}} \dots \psi^2\psi^1$$

is a preimage of x under $Sq^{2^{i+1}-1}$.

Proof. We fix a right \mathcal{A} -module M and the integer k . Proposition 3 implies $N \cap I_M(k) \subseteq N \cap \Delta_M(k)$. Now suppose $x \in N \cap \Delta_M(k)$. We shall prove that the elements $y_i = x\psi^{2^i}\psi^{2^{i-1}} \dots \psi^2\psi^1$ are in fact preimages of x under $Sq^{2^{i+1}-1}$ by first showing that for each $0 \leq j \leq i+1$,

$$y_i Sq^{2^{i+1}-1} = x\psi^{2^i}\psi^{2^{i-1}} \dots \psi^{2^{j+1}}\psi^{2^j} Sq^{2^j} Sq^{2^{j+1}} \dots Sq^{2^{i-1}} Sq^{2^i}, \quad (3)$$

where we understand this to mean $y_i Sq^{2^{i+1}-1} = x$ in the case $j = i+1$. The verification of Eqn. (3) is by induction on j . When $j = 0$, Eqn. (3) simply follows by definition of y_i and relations within \mathcal{A} :

$$\begin{aligned} y_i Sq^{2^{i+1}-1} &= x\psi^{2^i}\psi^{2^{i-1}} \dots \psi^2\psi^1 Sq^{2^{i+1}-1} \\ &= x\psi^{2^i}\psi^{2^{i-1}} \dots \psi^2\psi^1 Sq^1 Sq^2 \dots Sq^{2^{i-1}} Sq^{2^i} \end{aligned}$$

Next, suppose Eqn. (3) is verified for all numbers up to j for some $j \leq i$.

$$\begin{aligned} y_i Sq^{2^{i+1}-1} &= x\psi^{2^i} \dots \psi^{2^{j+1}}\psi^{2^j} Sq^{2^j} Sq^{2^{j+1}} \dots Sq^{2^i} \\ &= \left([x\psi^{2^i} \dots \psi^{2^{j+1}}] \psi^{2^j} Sq^{2^j} \right) Sq^{2^{j+1}} \dots Sq^{2^i} \\ &= \left([x\psi^{2^i} \dots \psi^{2^{j+1}}] Sq^{2^j} \psi^{2^j} + x\psi^{2^i} \dots \psi^{2^{j+1}} \right) Sq^{2^{j+1}} \dots Sq^{2^i} \\ &= \left([xSq^{2^j} \psi^{2^i} \dots \psi^{2^{j+1}}] \psi^{2^j} + x\psi^{2^i} \dots \psi^{2^{j+1}} \right) Sq^{2^{j+1}} \dots Sq^{2^i} \\ &= \left(0 + x\psi^{2^i} \dots \psi^{2^{j+1}} \right) Sq^{2^{j+1}} \dots Sq^{2^i} \\ &= x\psi^{2^i} \dots \psi^{2^{j+1}} Sq^{2^{j+1}} \dots Sq^{2^i} \end{aligned}$$

Note, we used the fact that $x \in N$, which is closed under each map $\psi^{2^i}, \psi^{2^{i-1}}, \dots, \psi^{2^j}$, in order to apply relations (1) and (2). We also used the fact that $x \in \Delta_M(k)$ to get $xSq^{2^j} = 0$. The upshot of Eqn. (3) is that when $j = i + 1$, it gives us exactly what we wanted:

$$y_i Sq^{2^{i+1}-1} = x.$$

Therefore, for each $0 \leq i \leq k$, we have $x \in \text{im} Sq^{2^{i+1}-1}$ with preimage y_i as specified in the statement of the theorem. \square

Remark 8. Though there may be many preimages for x , the elements y_i have the added property of being members of the null subspace N .

3. Shift Maps

In this section, we find certain k^{th} -order homotopy systems for various important right \mathcal{A} -modules. These homotopy systems are based on *shift maps*, which will be defined presently for elements of $\tilde{\Gamma}$. Recall, the generators of $\tilde{\Gamma}_{s,*}$ are s -length monomials in the non-commuting symbols $\{\gamma_i\}_{i \geq 1}$, where each $\gamma_i \in \tilde{H}_i(\mathbb{R}P^\infty)$ is the canonical generator. For convenience and readability in formulas, we generally denote a monomial $\gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_s} \in \tilde{\Gamma}_{s,*}$ by $[a_1, a_2, \dots, a_s]$. For each $s \geq 1, d \geq 0, r \geq 0$, and $1 \leq i \leq s$, define the i^{th} -place shift maps, $\psi_i^r : \tilde{\Gamma}_{s,d} \rightarrow \tilde{\Gamma}_{s,d+r}$ on generators by:

$$[a_1, a_2, \dots, a_s] \psi_i^r = [a_1, a_2, \dots, a_i + r, \dots, a_s].$$

We will find that (for a fixed i), the set of shifts $\{\psi_i^{2^m}\}_{m \leq k}$ forms a k^{th} -order homotopy system with a rather large null subspace. We start by showing there are certain commutation relations in $\tilde{\Gamma}_{1,*}$. The binomial coefficient $\binom{a}{b}$ is taken modulo 2, and unless otherwise stated, $\binom{a}{b} = 0$ if either $a < 0$ or $b < 0$. As a consequence of Lucas' Theorem,

$$\binom{a}{2^n} = \binom{b}{2^n}, \quad \text{if } a, b \geq 0 \text{ and } a \equiv b \pmod{2^{n+1}}. \quad (4)$$

Lemma 9. *For $m > n \geq 0$, if $a \geq 2^n$,*

$$[a] \psi_1^{2^m} Sq^{2^n} = [a] Sq^{2^n} \psi_1^{2^m}$$

Proof. Consider the expression on the left hand side:

$$[a] \psi_1^{2^m} Sq^{2^n} = [a + 2^m] Sq^{2^n} \quad (5)$$

$$= \binom{a + 2^m - 2^n}{2^n} [a + 2^m - 2^n]. \quad (6)$$

Since $m \geq n + 1$, $a + 2^m - 2^n \equiv a - 2^n \pmod{2^{n+1}}$. By hypothesis, $a - 2^n \geq 0$, and so by formula (4), there is equality $\binom{a + 2^m - 2^n}{2^n} = \binom{a - 2^n}{2^n}$.

$$[a]\psi_1^{2^m}Sq^{2^n} = \binom{a-2^n}{2^n}[a+2^m-2^n] \quad (7)$$

$$= \binom{a-2^n}{2^n}[a-2^n]\psi_1^{2^m} \quad (8)$$

$$= [a]Sq^{2^n}\psi_1^{2^m} \quad (9)$$

□

Corollary 10. *If $2^m > \ell \geq 0$ and $a \geq 2^{m-1}$,*

$$[a]\psi_1^{2^m}Sq^\ell = [a]Sq^\ell\psi_1^{2^m}.$$

Proof. Since $\ell < 2^m$, we have $Sq^\ell \in \mathcal{S}_{m-1}$. So Sq^ℓ can be written as a sum of products of “2-power” squares Sq^{2^i} such that $i < m$, each of which commutes with $\psi_1^{2^m}$ by Lemma 9. □

We will have occasion to use a more precise formula for the binomial coefficients:

$$\text{For } a \geq 0, \quad \binom{a}{2^n} = \begin{cases} 0, & \text{if } a \equiv 0, 1, \dots, 2^n - 1 \pmod{2^{n+1}}, \\ 1, & \text{if } a \equiv 2^n, 2^n + 1, \dots, 2^{n+1} - 1 \pmod{2^{n+1}}. \end{cases} \quad (10)$$

Lemma 11. *For $m \geq 0$, if $a \geq 2^m$,*

$$[a]\psi_1^{2^m}Sq^{2^m} + [a]Sq^{2^m}\psi_1^{2^m} = [a].$$

Proof. The proof is a straightforward computation:

$$\begin{aligned} & [a]\psi_1^{2^m}Sq^{2^m} + [a]Sq^{2^m}\psi_1^{2^m} \\ &= [a+2^m]Sq^{2^m} + \binom{a-2^m}{2^m}[a-2^m]\psi_1^{2^m} \end{aligned} \quad (11)$$

$$= \binom{a}{2^m}[a] + \binom{a-2^m}{2^m}[a]. \quad (12)$$

By formula (10), the expression in (12) is equal to $[a]$. □

These results will serve to prove that, under some conditions, the same commutation relations hold in $\tilde{\Gamma}_{s,*}$ for any $s \geq 1$.

Lemma 12. *If $2^m > \ell \geq 0$ and $a_i \geq 2^{m-1}$ in $x = [a_1, \dots, a_i, \dots, a_s]$, then*

$$x\psi_i^{2^m}Sq^\ell = xSq^\ell\psi_i^{2^m}.$$

Proof. Since Sq^ℓ commutes with the action of the symmetric group Σ_s , and there is also a relation, $\psi_i^r\sigma = \sigma\psi_{\sigma(i)}^r$, for $\sigma \in \Sigma_s$, it is sufficient to prove the

lemma for $i = 1$. Cor. 10 proves the case $s = 1$, so assume $s > 1$ and write $x = [a_1] \cdot z$, where $z = [a_2, \dots, a_s]$.

$$([a_1] \cdot z)Sq^\ell \psi_1^{2^m} = \sum_{p=0}^{\ell} ([a_1]Sq^p \cdot zSq^{\ell-p}) \psi_1^{2^m} \quad (13)$$

$$= \sum_{p=0}^{\ell} [a_1]Sq^p \psi_1^{2^m} \cdot zSq^{\ell-p} \quad (14)$$

$$= \sum_{p=0}^{\ell} [a_1]\psi_1^{2^m} Sq^p \cdot zSq^{\ell-p} \quad (15)$$

$$= ([a_1]\psi_1^{2^m} \cdot z)Sq^\ell \quad (16)$$

$$= ([a_1] \cdot z)\psi_1^{2^m} Sq^\ell. \quad (17)$$

Lines (14) and (17) follow from the fact that $\psi_1^{2^m}$ acts only on the first factor of a monomial. Line (15) follows from Cor. 10 since for each p in the sum, $p \leq \ell < 2^m$. \square

Lemma 13. For $m \geq 0$, if $a_i \geq 2^m$ in $x = [a_1, \dots, a_i, \dots, a_s]$, then

$$x\psi_i^{2^m} Sq^{2^m} + xSq^{2^m} \psi_i^{2^m} = x.$$

Proof. Again, it is sufficient to prove the lemma for $i = 1$ and $s > 1$. Write $x = [a_1] \cdot z$ for $z = [a_2, \dots, a_s]$. Consider the two terms separately:

$$\begin{aligned} x\psi_1^{2^m} Sq^{2^m} &= ([a_1] \cdot z)\psi_1^{2^m} Sq^{2^m} \\ &= ([a_1]\psi_1^{2^m} \cdot z)Sq^{2^m} \end{aligned} \quad (18)$$

$$= \sum_{p=0}^{2^m} [a_1]\psi_1^{2^m} Sq^p \cdot zSq^{2^m-p} \quad (19)$$

$$= \left(\sum_{p=0}^{2^m-1} [a_1]\psi_1^{2^m} Sq^p \cdot zSq^{2^m-p} \right) + [a_1]\psi_1^{2^m} Sq^{2^m} \cdot z. \quad (20)$$

$$\begin{aligned} xSq^{2^m} \psi_1^{2^m} &= ([a_1] \cdot z)Sq^{2^m} \psi_1^{2^m} \\ &= \sum_{p=0}^{2^m} ([a_1]Sq^p \cdot zSq^{2^m-p}) \psi_1^{2^m} \end{aligned} \quad (21)$$

$$= \sum_{p=0}^{2^m} [a_1]Sq^p \psi_1^{2^m} \cdot zSq^{2^m-p} \quad (22)$$

$$= \left(\sum_{p=0}^{2^m-1} [a_1]\psi_1^{2^m} Sq^p \cdot zSq^{2^m-p} \right) + [a_1]Sq^{2^m} \psi_1^{2^m} \cdot z. \quad (23)$$

In line (23) we just separate the sum into those terms for which $p < 2^m$ (so that Sq^p commutes with $\psi_1^{2^m}$), and the top term for which $p = 2^m$. Add lines (20) and (23), and note that the summations cancel completely:

$$x\psi_i^{2^m}Sq^{2^m} + xSq^{2^m}\psi_i^{2^m} = [a_1]\psi_1^{2^m}Sq^{2^m} \cdot z + [a_1]Sq^{2^m}\psi_1^{2^m} \cdot z \quad (24)$$

$$= \left([a_1]\psi_1^{2^m}Sq^{2^m} + [a_1]Sq^{2^m}\psi_1^{2^m}\right) \cdot z \quad (25)$$

$$= [a_1] \cdot z \quad (26)$$

$$= x. \quad (27)$$

Line (26) follows from Lemma 11 and completes the proof. \square

4. Homotopy Systems in some \mathcal{A} -modules

4.1. Homotopy Systems in $\tilde{\Gamma}$

The results of Lemmas 12 and 13 show that there is a homotopy system at work in $\tilde{\Gamma}$. Define for each $1 \leq i \leq s$, and $k \geq 0$, a graded subspace, $N_s(i, k) \subseteq \tilde{\Gamma}_{s,*}$,

$$N_s(i, k) = \text{span}\{[a_1, \dots, a_i, \dots, a_s] \in \tilde{\Gamma}_{s,*} \mid a_i \geq 2^k\}.$$

Proposition 14. *Fix $s \geq 1$ and $k \geq 0$. For each $1 \leq i \leq s$, the set of shift maps $\{\psi_i^{2^m}\}_{m \leq k}$ forms a k^{th} -order homotopy system of $\tilde{\Gamma}_{s,*}$ with null subspace $N_s(i, k)$.*

Proof. $N_s(i, k)$ is stable under ψ^{2^m} , since the effect of the shift map is to increase the index in position i . The verification of Eqns. (1) and (2) has been done on generators of $N_s(i, k)$ in Lemmas 12 and 13. \square

As an immediate corollary, we obtain:

Theorem 15. *Let $s \geq 1$ and $k \geq 0$. If $x \in N_s(i, k)$ for any $1 \leq i \leq s$, then $x \in \Delta(k)$ if and only if $x \in I(k)$.*

Proof. This follows from Prop. 14 and Thm. 7. \square

4.2. Localizations of $H^*(BV_s)$

Consider $H^*(BV_1, \mathbb{F}_2) \cong \mathbb{F}_2[t]$. The localization at the prime t , denoted Λ_1 , is an object studied in Adams [1], Singer [13], and elsewhere. Note that Λ_1 is a left \mathcal{A} -algebra, and so its graded vector space dual, $\nabla_1 = \Lambda_1^*$, is a right \mathcal{A} -algebra. Writing $[a] \in \nabla_1$ for the element dual to $t^a \in \Lambda_1$, we find the explicit action of the Steenrod squares on ∇_1 :

$$[a]Sq^i = \left(\binom{a-i}{i}\right) [a-i], \quad (28)$$

where the notation $\left(\binom{n}{i}\right)$ stands for the coefficient of x^i in the formal power series $(1+x)^n = \sum_{i \geq 0} \left(\binom{n}{i}\right) x^i$, for arbitrary $n \in \mathbb{Z}$.

Now for each $s \geq 1$, let ∇_s be the s -fold tensor product of ∇_1 with itself:

$$\nabla_s = (\nabla_1)^{\otimes s} \cong \text{span}\{[a_1, a_2, \dots, a_s] \mid a_i \in \mathbb{Z}\},$$

where the notation $[a_1, a_2, \dots, a_s]$ represents a monomial of non-commuting symbols, $[a_1] \otimes [a_2] \otimes \dots \otimes [a_s]$. Formula (28) is extended to elements of ∇_s via the Cartan formula so that the graded space $\nabla = \{\nabla_s\}_{s \geq 0}$ becomes a graded \mathcal{A} -algebra. We found that ∇_s has the remarkable property that it admits homotopy systems of all orders each of which has null subspace consisting of the entire space!

Theorem 16. *Let $s \geq 1$ and $k \geq 0$ and set $M = \nabla_s$. Then $\Delta_M(k) = I_M(k)$.*

Proof. Consider the shift system, $\{\psi_1^{2^m}\}_{m \leq k}$ defined in Section 3. The definition of $\psi_1^{2^m}$ can be extended to elements of ∇_s in the straightforward way, namely, $\psi_1^{2^m}$ increases the first index of $[a_1, \dots, a_s] \in \nabla_s$ by 2^m . Now Eqn. (10) generalizes neatly:

$$\text{For } a \in \mathbb{Z}, \quad \binom{a}{2^n} = \begin{cases} 0, & \text{if } a \equiv 0, 1, \dots, 2^n - 1 \pmod{2^{n+1}}, \\ 1, & \text{if } a \equiv 2^n, 2^n + 1, \dots, 2^{n+1} - 1 \pmod{2^{n+1}}. \end{cases} \quad (29)$$

Since there is now no restriction on the value of a , we obtain unrestricted relations in ∇_1 . For any $a \in \mathbb{Z}$,

$$\begin{aligned} [a]\psi_1^{2^m}Sq^\ell &= [a]Sq^\ell\psi_1^{2^m}, & \text{if } 2^m > \ell \geq 0 \\ [a]\psi_1^{2^m}Sq^{2^m} + [a]Sq^{2^m}\psi_1^{2^m} &= [a] \end{aligned}$$

The relations are easily shown on ∇_s in general. For any $x \in \nabla_s$,

$$\begin{aligned} x\psi_1^{2^m}Sq^\ell &= xSq^\ell\psi_1^{2^m}, & \text{if } 2^m > \ell \geq 0 \\ x\psi_1^{2^m}Sq^{2^m} + xSq^{2^m}\psi_1^{2^m} &= x \end{aligned}$$

Thus there is a k^{th} -order homotopy system on ∇_s with null subspace the entire space ∇_s . \square

4.3. Homotopy Systems in $\mathbb{F}_2 \otimes_G \tilde{\Gamma}_{s,*}$

For fixed $s \geq 1$, consider a subgroup G of $GL(s, \mathbb{F}_2)$. The \mathcal{A}^+ -annihilated problem in $\mathbb{F}_2 \otimes_G \tilde{\Gamma}_{s,*}$ is one of current interest, being dual to the hit problem for the subspace $H^*(\prod^s \mathbb{R}P^\infty, \mathbb{F}_2)^G$ of G -invariant elements [7, 14]. When G is a non-trivial subgroup of $GL(s, \mathbb{F}_2)$, the i^{th} -place shift maps $\{\psi_i^{2^m}\}$ defined for $\tilde{\Gamma}_{s,*}$ may not be well-defined on $\mathbb{F}_2 \otimes_G \tilde{\Gamma}_{s,*}$. However, certain results can still be found. First consider the case $G = \Sigma_s$. Denote by $\langle a_1, a_2, \dots, a_s \rangle$ the image of $[a_1, a_2, \dots, a_s]$ under the canonical projection $\tilde{\Gamma}_{s,*} \rightarrow \mathbb{F}_2 \otimes_{\Sigma_s} \tilde{\Gamma}_{s,*}$. Note, the a_i 's may freely change positions in the expression $\langle a_1, a_2, \dots, a_s \rangle$. We make the convention that any such term has $a_1 \geq a_2 \geq \dots \geq a_s$.

Theorem 17. *Let $s \geq 1$ and $k \geq 0$, and set $M = \mathbb{F}_2 \otimes_{\Sigma_s} \tilde{\Gamma}_{s,*}$. Suppose $x \in M$ is a sum of terms of the form $\langle a_1, a_2, \dots, a_s \rangle$ such that $a_1 - a_2 \geq 2^k$. Then $x \in \Delta_M(k)$ if and only if $x \in I_M(k)$.*

Proof. We first define a homotopy system on $\mathbb{F}_2 \otimes_{\Sigma_s} \tilde{\Gamma}_{s,*}$. For $m \leq k$, define ψ^{2^m} on generators:

$$\langle a_1, a_2, \dots, a_s \rangle \psi^{2^m} = \langle a_1 + 2^m, a_2, \dots, a_s \rangle$$

The map is well-defined because of the convention that $a_1 \geq a_2 \geq \dots \geq a_s$. Now since $a_1 \geq 2^k + a_2$, Lemmas 12 and 13 should apply to the shift map we have defined, but there is a subtlety here. After applying Sq^{2^m} , the values can decrease (and hence be permuted so as to maintain the convention of writing the values in non-increasing order), but if $a_1 - a_2 \geq 2^k$, then all terms of $\langle a_1, a_2, \dots, a_s \rangle Sq^{2^m}$ (for $m \leq k$) are of the form $\langle b_1, b_2, \dots, b_s \rangle$, where $b_1 \geq a_2 \geq b_2, b_3, \dots, b_s$, and so the shift operation will act on the correct index. This dictates that the appropriate null subspace is the span of elements of the form $\langle a_1, a_2, \dots, a_s \rangle$ such that $a_1 - a_2 \geq 2^k$. Clearly, this space is invariant under the shift operations ψ^{2^m} as defined in this proof. \square

A similar trick can be used when $G = C_s$, the cyclic group of order s . Denote by (a_1, a_2, \dots, a_s) the image of $[a_1, a_2, \dots, a_s]$ under the canonical projection $\tilde{\Gamma}_{s,*} \rightarrow \mathbb{F}_2 \otimes_{C_s} \tilde{\Gamma}_{s,*}$. In the term (a_1, a_2, \dots, a_s) , the values may be cyclically permuted, and so generators of $\mathbb{F}_2 \otimes_{C_s} \tilde{\Gamma}_{s,*}$ can be taken to have the form (a_1, a_2, \dots, a_s) where a_1 is a maximal value in the list. There is still some ambiguity as there could be multiple maximal values in the list $\{a_i\}$, and so we need to make a definitive choice in such cases. Although the exact choice we make will be immaterial in the following arguments, we nevertheless must specify a consistent convention. Let $m = \max\{a_i\}$, and choose the cyclic order of $\{a_i\}$ (given by an element $g \in C_s$) such that when $(a_{g(1)}, a_{g(2)}, \dots, a_{g(s)})$ is read left-to-right as a number in base $m+1$, that number is maximal among all such numbers for various cyclic orders of the $\{a_i\}$.

Theorem 18. *Let $s \geq 1$ and $k \geq 0$, and set $M = \mathbb{F}_2 \otimes_{C_s} \tilde{\Gamma}_{s,*}$. Suppose $x \in M$ is a sum of terms of the form (a_1, a_2, \dots, a_s) such that for all $i > 1$, we have $a_1 - a_i > 2^k$. Then $x \in \Delta_M(k)$ if and only if $x \in I_M(k)$.*

Proof. Use the homotopy system $\{\psi_1^{2^m}\}_{m \leq k}$, defined originally for $\tilde{\Gamma}_{s,*}$ (applied to generators of M written in the order specified above). The requirement that a_1 is at least 2^k greater than any other a_i ensures that the squaring operations do not “mix up” the order of factors in each term. \square

5. The Quotient $\Delta_M(k)/I_M(k)$

For any right \mathcal{A} -module M , $I_M(k)$ is a vector subspace of $\Delta_M(k)$. However, when M is an \mathcal{A} -algebra, much more can be said. The following was observed by Singer [12].

Proposition 19. *If M is a right \mathcal{A} -algebra, then for each $k \geq 0$, $I_M(k)$ is a two-sided ideal of the algebra $\Delta_M(k)$.*

Proof. The fact that $\Delta_M(k)$ is an algebra follows easily from the Cartan formula. Now let $x \in I_M(k)$ and $y \in \Delta_M(k)$. For each $i \leq k$, write $x = z_i Sq^{2^{i+1}-1}$. Note, for $i = 0, 1, 2, \dots, k$, $Sq^{2^{i+1}-1} \in \mathcal{S}_k^+$. Therefore, since y is \mathcal{S}_k^+ -annihilated, we have for each $i \leq k$:

$$yx = y \cdot z_i Sq^{2^{i+1}-1} = (yz_i) Sq^{2^{i+1}-1},$$

which proves $yx \in I_M(k)$. Similarly, $xy \in I_M(k)$. \square

Definition 20. For each $k \geq 0$, the space of *un-hit* \mathcal{S}_k^+ -annihilateds is the quotient of spaces,

$$U_M(k) = \Delta_M(k)/I_M(k).$$

If M is graded, then $U_M(k)$ inherits this grading. If M is an \mathcal{A} -algebra, then $U_M(k)$ is an algebra with product induced from the product in M .

Remark 21. When M is the homology of an associative H -space, then the Pontryagin product makes M into a right \mathcal{A} -algebra, and hence each $U_M(k)$ is an algebra. On the other hand, if $M = \{M_{s,*}\}_{s \geq 0}$ is bigraded with $M_{s,*} = H_*(X_s, \mathbb{F}_2)$, where $\{X_s\}_{s \geq 0}$ is a family of spaces equipped with an associative pairing $X_s \times X_t \rightarrow X_{s+t}$, then M admits the structure of bigraded right \mathcal{A} -algebra, and so each $U_M(k)$ is a bigraded algebra. It is interesting to see that the bigraded $\tilde{\Gamma} = \{\tilde{\Gamma}_{s,*}\}_{s \geq 0}$ possesses both types of multiplication, and so $U_{\tilde{\Gamma}}(k)$ is a right \mathcal{A} -algebra in two very different ways!

6. Fine Structure of $\Delta(1)$

For the remainder of this paper, we examine the structure of $\Delta(1) = \ker Sq^1 \cap \ker Sq^2 \subseteq \tilde{\Gamma}$. We will eventually produce an element $z \in \Delta(1)_{5,9}$ that is not in the image of Sq^3 , hence not in $I(1)_{5,9}$. In this section, fix $s \geq 2$, $d \geq s$, and write $x \in \tilde{\Gamma}_{s,d}$ in the form $x = \sum_{i \geq 1} [i] \cdot x_i$ for elements $x_i \in \tilde{\Gamma}_{s-1,d-i}$. The following two propositions characterize what it means for x to be in $\ker Sq^1$, *resp.* $\ker Sq^2$.

Proposition 22. *If $x = \sum_{i \geq 1} [i] \cdot x_i \in \ker Sq^1$, then for each $n \geq 1$,*

$$x_{2n} = x_{2n-1} Sq^1, \quad \text{and} \tag{30}$$

$$x_{2n} Sq^1 = 0. \tag{31}$$

Remark 23. Of course, Eqn. (31) follows as a consequence of Eqn. (30).

Proof. Apply Sq^1 to both sides of $x = \sum_{i \geq 1} [i] \cdot x_i$.

$$\begin{aligned}
0 &= \left(\sum_{i \geq 1} [i] \cdot x_i \right) Sq^1 \\
&= \sum_{n \geq 1} [2n-1] \cdot x_{2n-1} Sq^1 + \sum_{n \geq 1} ([2n-1] \cdot x_{2n} + [2n] \cdot x_{2n} Sq^1) \\
&= \sum_{n \geq 1} [2n-1] \cdot (x_{2n-1} Sq^1 + x_{2n}) + \sum_{n \geq 1} [2n] \cdot x_{2n} Sq^1.
\end{aligned}$$

Since $\tilde{\Gamma}_{s,*}$ is a free algebra on the generators $\{[i]\}_{i \geq 1}$, the above computation proves (30) and (31). \square

Proposition 24. *If $x = \sum_{i \geq 1} [i] \cdot x_i \in \ker Sq^2$, then for each $m \geq 1$,*

$$x_{4m-2} Sq^1 = x_{4m-3} Sq^2, \quad (32)$$

$$x_{4m} = x_{4m-2} Sq^2, \quad (33)$$

$$x_{4m+1} = x_{4m-1} Sq^2 + x_{4m} Sq^1, \quad \text{and} \quad (34)$$

$$x_{4m} Sq^2 = 0. \quad (35)$$

Proof. Apply Sq^2 to both sides of $x = \sum_{i \geq 1} [i] \cdot x_i$.

$$0 = \left(\sum_{i \geq 1} [i] \cdot x_i \right) Sq^2$$

We then expand using the Cartan formula and perform many tedious (though elementary) manipulations on the summation. The details are left to the diligent reader. Care must be taken with the low order terms, but eventually we find:

$$\begin{aligned}
0 &= [1] \cdot (x_1 Sq^2 + x_2 Sq^1) + [2] \cdot (x_2 Sq^2 + x_4) + [3] \cdot (x_3 Sq^2 + x_4 Sq^1 + x_5) \\
&\quad + \sum_{m \geq 1} [4m] \cdot x_{4m} Sq^2 \\
&\quad + \sum_{m \geq 1} [4m+1] \cdot (x_{4m+1} Sq^2 + x_{4m+2} Sq^1) \\
&\quad + \sum_{m \geq 1} [4m+2] \cdot (x_{4m+2} Sq^2 + x_{4m+4}) \\
&\quad + \sum_{m \geq 1} [4m+3] \cdot (x_{4m+3} Sq^2 + x_{4m+4} Sq^1 + x_{4m+5})
\end{aligned}$$

Including each of the low order terms into the appropriate summation, the individual summations then give each of the relations, (32), (33), (34), and (35). \square

Proposition 25. *Suppose $x \in \tilde{\Gamma}_{s,*}$ for some $s \geq 2$. Then $x \in \Delta(1)$ if and only if $x = \sum_{i \geq 1} [i] \cdot x_i$, where the $x_i \in \tilde{\Gamma}_{s-1,*}$ satisfy:*

$$x_1 \in \ker Sq^2, \quad (36)$$

$$x_2 = x_1 Sq^1, \quad (37)$$

$$x_3 \in (Sq^1)^{-1}(x_1 Sq^3), \quad (38)$$

$$\text{and for each } m \geq 1, \quad x_{4m} = x_{4m-1} Sq^1, \quad (39)$$

$$x_{4m+1} = x_{4m-1} Sq^2, \quad (40)$$

$$x_{4m+2} = x_{4m-1} Sq^2 Sq^1, \quad (41)$$

$$x_{4m+3} \in (Sq^1)^{-1}[x_{4m-1} Sq^2 Sq^3]. \quad (42)$$

Proof. Fix $s \geq 2$, and let $x \in \Delta(1)_{s,*}$. Then $x Sq^1 = x Sq^2 = 0$. Prop. 22 gives $x_2 = x_1 Sq^1$ and for each $m \geq 1$, $x_{4m} = x_{4m-1} Sq^1$, proving (37) and (39). Next, with $m = 1$, (32) gives $x_1 Sq^2 = x_2 Sq^1$. Then (30) can be used: $x_2 Sq^1 = (x_1 Sq^1) Sq^1 = 0$, proving (36). In order to prove (38), all that we must show is that $x_3 Sq^1 = x_1 Sq^3$. This is done by using (33) with $m = 1$: $x_4 = x_2 Sq^2$, and then applying (30) to get: $x_3 Sq^1 = x_1 Sq^1 Sq^2 = x_1 Sq^3$. Next, (34) together with (31) yields (40). Apply Sq^1 to both sides of (40) to get (41). Finally, to prove (42), observe (for any $m \geq 1$):

$$\begin{aligned} x_{4m+4} &= x_{4m+2} Sq^2, \quad \text{by (33)} \\ x_{4m+3} Sq^1 &= x_{4m+1} Sq^1 Sq^2, \quad \text{by (30)} \\ x_{4m+3} Sq^1 &= x_{4m-1} Sq^2 Sq^1 Sq^2, \quad \text{by (40)} \\ &= x_{4m-1} Sq^2 Sq^3. \end{aligned}$$

This concludes the forward direction of the proof.

For the reverse direction, we are given $x \in \tilde{\Gamma}_{s,*}$ with $s \geq 2$, and $x = \sum_{i \geq 1} [i] \cdot x_i$, with the relations on x_i as stated in Prop. 25. We must show that $x Sq^1 = 0$ and $x Sq^2 = 0$. This routine verification is omitted. \square

Corollary 26. *The elements $x = \sum_{i \geq 1} [i] \cdot x_i \in \Delta(1)$ are determined by a choice of $x_1 \in \ker Sq^2$, a choice of $x_3 \in (Sq^1)^{-1}[x_1 Sq^3]$, and for each $m \geq 1$, a choice of $x_{4m+3} \in (Sq^1)^{-1}[x_{4m-1} Sq^2 Sq^3]$.*

Remark 27. By Prop. 25, $x \in \Delta(1)$ if and only if $x_1 \in \ker Sq^2$ and certain other relations hold on other x_i . Corollary 26 states that any choice of $x_1 \in \ker Sq^2$ can be built up into an element $x \in \Delta(1)$, or in other words, the choice of $x_1 \in \ker Sq^2$ is not restricted by any other relation.

7. What is in $U_M(k)$?

The machinery of homotopy systems provides a way to categorize large subspaces of \mathcal{S}_k^+ -annihilated elements that are in the image of spike squares. In a sense, the elements of $I_M(k)$ are easy to handle, and so our attention turns to those \mathcal{S}_k^+ -annihilateds that are not in $I_M(k)$. In other words,

Question 28. For a given right \mathcal{A} -module M , what is in $U_M(k)$?

Example 29. In the case $M = \nabla_s$, Thm. 16 shows $\Delta_M(k) = I_M(k)$ for all k . In other words, $U_{\nabla}(k) = 0$ for all $k \geq 0$. However, this seems to be an extreme case.

Example 30. $U_{\tilde{\Gamma}}(0) = \mathbb{F}_2$, concentrated in bidegree $(0, 0)$, as a consequence of Thm. 15.

In any unstable right \mathcal{A} -module M , there are a number of elements of $\Delta_M(k)$ that are un-hit (*i.e.* fail to be in $I_M(k)$) only because their degree is so low that the instability condition prevents their being in the image of higher order squares. Let us ignore those types of un-hit elements as *degenerate*; such elements should be considered in $\Delta_M(j)$ for some $j < k$. It is much more interesting to know if there are un-hit elements in high degrees.

Example 31. It is an easy exercise to show that for $k \geq 1$, there are only degenerate elements in $U_{\tilde{\Gamma}}(k)_{1,*}$. In particular, if $d \geq 2^{k+1}$ then $U_{\tilde{\Gamma}}(k)_{1,d} = 0$.

Proposition 32. Let $x = \sum_{i \geq 1} [i] \cdot x_i \in \Delta(1)$. Then $x \in I(1)$ if and only if $x_1 = wSq^2$ for some $w \in \ker Sq^3$.

Proof. Let $x \in \Delta(1)_{s,d}$, for some $s \geq 2$. It suffices to prove: $x \in \text{im } Sq^3$ if and only if $x_1 = wSq^2$ for some $w \in \ker Sq^3$. Write $x = [1] \cdot x_1 + x'$. I claim $x' \in \text{im } Sq^3$; indeed, there is a preimage:

$$y' = \sum_{j \geq 2} [2j] \cdot x_{2j-3}.$$

It is straightforward to verify that $y'Sq^3 = x'$, using Prop. 25. Thus, it suffices to prove: $[1] \cdot x_1 \in \text{im } Sq^3$ if and only if $x_1 = wSq^2$ for some $w \in \ker Sq^3$.

A Sq^3 -preimage of $[1] \cdot x_1$ could only have the form $[1] \cdot y_1 + [2] \cdot y_2$ for some $y_1, y_2 \in \tilde{\Gamma}_{s-1,*}$. Consider the effect of applying Sq^3 on this element:

$$\begin{aligned} ([1] \cdot y_1 + [2] \cdot y_2) Sq^3 &= [1] \cdot (y_1 Sq^3 + y_2 Sq^2) + [2] \cdot y_2 Sq^3 \\ &= [1] \cdot (y_1 Sq^1 + y_2) Sq^2 + [2] \cdot y_2 Sq^3. \end{aligned}$$

Thus we find that for $w = y_1 Sq^1 + y_2$, $[1] \cdot x_1 \in \text{im } Sq^3$ implies $wSq^2 = x_1$, and $wSq^3 = y_2 Sq^3 = 0$. Conversely, if any w exists such that $wSq^2 = x_1$ and $wSq^3 = 0$, then $([2] \cdot w) Sq^3 = [1] \cdot x_1$. \square

Corollary 33. There are only degenerate elements in $U_{\tilde{\Gamma}}(1)_{2,*}$.

Proof. By Cor. 26 all elements of $\Delta(1)_{2,*}$ are determined by a choice of $x_1 \in \ker Sq^2$, together with choices for x_{4m-1} , $m \geq 1$. By Prop. 32, it suffices to examine only the first term $[1] \cdot x_1$ of an element $x \in \Delta(1)$. Now $x_1 \in \tilde{\Gamma}_{1,*}$, and it is easy to see that $\ker Sq^2 = \text{im } Sq^2$ in $\tilde{\Gamma}_{1,d}$ for $d \geq 2$. Furthermore, all preimages of Sq^2 in $\tilde{\Gamma}_{1,*}$ (namely, elements of the form $[4j]$ and $[4j+1]$ for $j \geq 1$), are killed by Sq^3 . \square

Note that Theorem 7 can only be used to show particular elements $x \in \Delta_M(k)$ are trivial in $U_M(k)$, not to detect non-trivial elements, and so it may be very difficult to understand the true nature of $U_M(k)$. Initially we found no non-trivial non-degenerate elements of $U_{\tilde{\Gamma}}(k)$. However, using Prop. 25, Cor. 26, and Prop. 32, together with machine computations, a non-trivial element $\bar{z} \in U_{\tilde{\Gamma}}(1)_{5,9}$ can be produced.

Corollary 34. *Define:*

$$\begin{aligned} z = & [1, 1, 1, 2, 4] + [1, 1, 2, 1, 4] + [1, 1, 2, 4, 1] + [1, 2, 1, 4, 1] + [1, 2, 2, 2, 2] \\ & + [1, 4, 1, 1, 2] + [1, 4, 2, 1, 1] + [2, 1, 1, 2, 3] + [2, 1, 2, 1, 3] + [2, 1, 2, 2, 2] \\ & + [2, 1, 2, 3, 1] + [2, 2, 1, 2, 2] + [2, 2, 1, 3, 1] + [2, 2, 2, 1, 2] + [2, 2, 2, 2, 1] \\ & + [2, 3, 1, 1, 2] + [2, 3, 2, 1, 1] + [3, 1, 2, 1, 2] + [3, 1, 2, 2, 1] + [3, 2, 2, 1, 1] \\ & + [4, 1, 1, 1, 2] + [4, 1, 1, 2, 1] + [4, 1, 2, 1, 1] + [4, 2, 1, 1, 1] + [5, 1, 1, 1, 1]. \end{aligned}$$

The element $z \in \tilde{\Gamma}_{5,9}$ represents a non-trivial element of $U_{\tilde{\Gamma}}(1)$.

Proof. Let $w = [1, 1, 2, 4] + [1, 2, 1, 4] + [1, 2, 4, 1] + [2, 1, 4, 1] + [2, 2, 2, 2] + [4, 1, 1, 2] + [4, 2, 1, 1]$. It can be verified by hand that $wSq^2 = 0$ and by computer that $w \notin \text{im } Sq^2$. By Prop. 25 and Cor. 26, $w \in \ker Sq^2$ can be used to produce an element $z = \sum_{i \geq 1} [i] \cdot z_i \in \Delta(1)$ with $z_1 = w$. Such element z cannot be in the image of Sq^3 due to Prop. 32 (a fact that can be independently checked by computer as well). \square

Remark 35. The proof of Cor. 34 hinges on the existence of an element $w \in \ker Sq^2 \setminus \text{im } Sq^2$. Even though Sq^2 does not act as a differential on $\tilde{\Gamma}_{s,*}$ for $s \geq 2$, it is interesting to see how closely related $\ker Sq^2$ and $\text{im } Sq^2$ really are. We do not expect $\text{im } Sq^2 \subseteq \text{im } Sq^2$, however it is hard to find counterexamples to $\ker Sq^2 \subseteq \text{im } Sq^2$. The author hopes to explore this relationship in future work.

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